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On the next-to-leading holographic entanglement entropy in AdS_3/CFT_2

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ABSTRACT: We reconsider the one-loop correction to the holographic entanglement entropy in AdS_3/CFT_2 by analysing the contributions due to a bulk higher spin s current or a scalar field with scaling dimension Δ . We consider the two-interval case and work perturbatively in their small cross ratio x . We provide various results for the entanglement entropy due to the so-called CDW elements of the associated Schottky uniformization group. In particular, in the higher spin current case, we obtain a closed formula for all the contributions of the form $\mathcal{O}(x^{2s+p})$ up to $\mathcal{O}(x^{4s})$, where 2-CDW elements are relevant. In the scalar field case, we calculate the similar contributions for generic values of Δ . The terms up to $\mathcal{O}(x^{2\Delta+5})$ are compared with an explicit CFT calculation with full agreement. The analysis exploits various simplifications which are valid in the strict entanglement limit of the Rényi entropy. This allows to identify in a clean way the relevant operators that provide the gravity result. The 2-CDW contributions are also analysed and a closed formula for the leading $\mathcal{O}(x^{4s})$ coefficient is presented as a function of the generic spin s . As a specific application, we combine the CDW and 2-CDW calculations and present the complete $\mathcal{O}(x^{4s+2})$ entanglement entropy for a spin $s = 2, 3, 4$ higher spin currents.

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1 Introduction and summary of results

A quite general feature of quantum gravity is the holographic principle. Roughly speaking, it states that the number of degrees of freedom of a $(d + 2)$ -dimensional quantum gravity theory is comparable to that of a non-gravitational quantum system in $d + 1$ dimensions [1, 2]. This remarkable feature of gravity theories is one of the most physically intriguing aspects of the Bekenstein-Hawking formula [3, 4] for the entropy S_{BH} of a black hole. Indeed, S_{BH} turns out to be proportional to the area of the black hole event horizon instead of its volume. In recent times, the powerful framework of AdS/CFT correspondence [5] allowed to study several theoretical laboratories with explicit examples of theories where quantum gravity on $(d + 2)$ -dimensional anti-de Sitter spacetime AdS_{d+2} is holographically equivalent to a conformal field theory CFT_{d+1} .

A very interesting open problem in AdS/CFT is to characterise the region of the AdS bulk space that encodes the information associated with the dual conformal theory [6, 7]. From this perspective, a promising quantity to answer this question is the entanglement entropy. It is a universal (non-local) observable in the sense that its definition does not depend on the microscopic details of the theory under consideration. Also, it can be defined in full generality for a quantum system. Indeed, it is simply the von Neumann entropy S_A associated with the reduced density matrix obtained by integrating out the degrees of freedom inside a d -dimensional space-like subspace B , the complement of A , of a given $(d + 1)$ - dimensional quantum field theory. The entanglement entropy S_A measures how the subsystems A and B are correlated with each other and has some analogy with black hole entropy. The entanglement entropy is expected to be related to the number of degrees of freedom. Indeed, in two-dimensional conformal field theory, general arguments show that it is proportional to the central charge c [8]. Finally, as opposed to the usual thermal entropy, the entanglement entropy does not vanish at zero temperature and therefore it can probe the properties of the ground state of a given quantum system.

A remarkable AdS/CFT result for the calculation of the entanglement entropy is the celebrated Ryu-Takayanagi formula [6, 9]. It states that, at leading order in the semiclassical expansion,

$$S_A = \frac{\text{Area}(\Sigma_A)}{4 G_N^{(d+2)}}, \quad (1.1)$$

where Σ_A is the d -dimensional minimal surface whose boundary is the $(d - 1)$ -dimensional manifold $\partial\Sigma_A = \partial A$, the interface between A and B , and the constant $G_N^{(d+2)}$ is Newton's constant in AdS_{d+2} . The Ryu-Takayanagi formula can be tested by quantum field theoretical calculations in the dual conformal theory. At leading order in the $1/c$ expansion [10–12] one finds that when A is a set of disjoint intervals with lengths ℓ_i , the entanglement entropy is rigorously

$$S_A = \frac{c}{3} \sum_i \log \frac{\ell_i}{a} + \mathcal{O}(c^0), \quad (1.2)$$

where a is an ultraviolet small length scale. This scaling form is in agreement with old calculations [8, 13, 14]. Also, it can be shown that this expression agrees with the Ryu-Takayanagi formula, using the AdS/CFT relation $\ell_{AdS}/G_N = 2c/3$, see [10].

The actual calculation of S_A is done by the replica trick that amounts to the calculation of the $n \rightarrow 1$ limit of the n -th order Rényi entropy [15]

$$S_A = - \lim_{n \rightarrow 1} \frac{\partial}{\partial n} \text{Tr}_A \rho_A^n, \quad (1.3)$$

where ρ_A is the reduced density matrix and A is a disjoint union of intervals. The trace is computed by evaluating the partition function Z_n on a n -sheeted Riemann surface with branch cuts at the endpoints of A as

$$\text{Tr}_A \rho_A^n = \frac{Z_n}{Z_1^n}. \quad (1.4)$$

This complicated ratio can be evaluated directly in CFT by introducing suitable twist operators to implement the branch cuts. Alternatively, it can be computed by exploiting AdS/CFT.

The analysis of the subleading corrections $\mathcal{O}(c^0)$ has been initiated in [16]. In the semiclassical expansion of the partition function at large central charge, the leading correction to the on-shell gravity action is expressed in terms of functional determinants of the operators describing quadratic fluctuations of all the bulk fields. This one-loop correction is perturbatively exact in pure three dimensional gravity [17]. An explicit elegant expression for functional determinants on quotients of AdS_3 by a Schottky group has been obtained in [18, 19].

Using these tools, the one-loop entanglement entropy for the two-interval case has been computed perturbatively at small cross ratio between the four endpoints in [16]. The authors of this paper also considered the case of a single interval on a circle at finite temperature. In particular, they found that, at high temperatures, the one-loop contributions introduce finite size corrections to the entanglement entropy that are not present classically, see also [20]. Recently, the AdS/CFT matching of the entanglement entropy has been extensively tested in the series of papers [21–23]. In particular, the holographic Rényi entropy has been computed in higher spin AdS_3 gravity with additional \mathcal{W}_3 or \mathcal{W}_4 symmetry at order $\mathcal{O}(x^8)$ in the small cross ratio x for the case of two intervals. The predictions from gravity and CFT perfectly agree and the explicit results for a spin s higher spin current are

$$\begin{aligned} S^{(s=2)}_{\text{one-loop}} &= -\frac{x^4}{630} - \frac{2x^5}{693} - \frac{15x^6}{4004} - \frac{x^7}{234} - \frac{167x^8}{36036} + \mathcal{O}(x^9), \\ S^{(s=3)}_{\text{one-loop}} &= -\frac{x^6}{12012} - \frac{x^7}{4290} - \frac{7x^8}{16830} + \mathcal{O}(x^9), \\ S^{(s=4)}_{\text{one-loop}} &= -\frac{x^8}{218790} + \mathcal{O}(x^{19}). \end{aligned} \tag{1.5}$$

Similar calculations has been performed at order $\mathcal{O}(x^6)$ in various massive gravities at the critical points.

Recently, in [24], the Rényi entropy contributions of Virasoro primaries with arbitrary scaling weights has been computed to leading and next-to-leading order in the interval size. When the primaries are higher spin currents, such terms are placed in one-to-one correspondence with terms in the bulk 1-loop determinants for higher spin gauge fields propagating on handlebody geometries. Deformation by chemical potentials associated with higher spin currents has been very recently addressed in [25] with promising results.

An interesting missing piece in this series of checks are bulk scalars. A motivation for their study is in the recent discovery that Vasiliev higher spin theory on AdS_3 [26, 27] coupled to a complex scalar is dual to a class of coset WZW conformal theories in certain large- N limit [28].

In this paper, we elaborate on the one-loop corrections to the entanglement entropy. Our main results are explicit closed formulae for the entanglement entropy due to a higher

spin current as well as accurate series expansions for the contribution from scalars. We work in the two-interval case and in the regime where their cross ratio x is a small perturbative parameter. Technically, the contributions to the entanglement entropy can be written as sums over Schottky group elements γ and organised according to the complexity of γ . In more details, one can consider specific classes of elements γ known as N -CDW elements. For a higher spin s current, the contribution to the entropy from each class starts at order $\mathcal{O}(x^{2Ns})$. The same holds for a scalar field with its scaling dimension Δ playing the role of s .

We analyse in great detail the CDW contributions ($N = 1$) for both the higher spin currents and a scalar field. In particular, we derive, in the current case, the following exact nice formula

$$S^{(s) \text{ one-loop}} = -\frac{\sqrt{\pi}}{2} \left(\frac{x}{4}\right)^{2s} \frac{\Gamma(2s+1)}{\Gamma(2s+\frac{3}{2})} {}_3F_2\left(2s, 2s-\frac{1}{2}, 2s+1; 2s+\frac{3}{2}, 4s-1; x\right) + \mathcal{O}(x^{4s}). \quad (1.6)$$

where we have explicitly indicated that the formula holds for all terms up to $\mathcal{O}(x^{4s-1})$. At order $\mathcal{O}(x^{4s})$ other CDW contributions modify this formula together with 2-CDW terms. In the paper, we discuss several series expansions for these further contributions and we give them parametrically in s as far as possible. One remarkable result is a closed formula for the leading 2-CDW contribution that reads

$$S_{2\text{-CDW}}^{(s) \text{ one-loop}} = \sigma_s x^{4s} + \mathcal{O}(x^{4s+1}), \quad (1.7)$$

where the coefficient σ_s is obtained from

$$\sigma_s = -\frac{1}{2} \left[\frac{1}{\Gamma(2s+1)^4} (p-s)_{2s}^4 - \frac{1}{\Gamma(4s+1)^2} (p-2s)_{4s}^2 \right]_{p^{N \rightarrow -N} \zeta(1-N)}, \quad (1.8)$$

where $(a)_b$ is a Pochhammer symbol.

To give an idea of the accuracy allowed by the tools developed in this paper, we improved the known expansions (1.5) by extending them up to the order $\mathcal{O}(x^{4s+2})$ with the explicit result

$$\begin{aligned} S^{(s=2) \text{ one-loop}} &= -\frac{x^4}{630} - \frac{2x^5}{693} - \frac{15x^6}{4004} - \frac{x^7}{234} - \frac{167x^8}{36036} - \frac{69422x^9}{14549535} - \frac{122x^{10}}{24871} + \mathcal{O}(x^{11}), \\ S^{(s=3) \text{ one-loop}} &= -\frac{x^6}{12012} - \frac{x^7}{4290} - \frac{7x^8}{16830} - \frac{28x^9}{46189} - \frac{15x^{10}}{19019} - \frac{2x^{11}}{2093} - \frac{1644627x^{12}}{1487285800} \\ &\quad - \frac{458893x^{13}}{371821450} - \frac{224484047x^{14}}{166363540200} + \mathcal{O}(x^{15}), \\ S^{(s=4) \text{ one-loop}} &= -\frac{x^8}{218790} - \frac{4x^9}{230945} - \frac{3x^{10}}{76076} - \frac{5x^{11}}{71162} - \frac{11x^{12}}{101660} \\ &\quad - \frac{11x^{13}}{72675} - \frac{1001x^{14}}{5058180} - \frac{143x^{15}}{580754} - \frac{253702367x^{16}}{859544957700} \\ &\quad - \frac{1550029508x^{17}}{4512611027925} - \frac{207186247x^{18}}{530052723915} + \mathcal{O}(x^{19}). \end{aligned} \quad (1.9)$$

The case of a scalar field is also considered in great detail. In this case, we also present a detailed discussion of the matching with a direct CFT calculation. In this way, we reproduce the gravity calculation of the entropy at order up to $\mathcal{O}(x^{2\Delta+5})$. This comparison is not surprising, but is somewhat interesting in itself due to various simplifications that we emphasise in the strict entanglement limit $n \rightarrow 1$.

The plan of the paper is the following. In Sec. (2), we briefly recall the technical tools that are needed in order to compute the holographic entanglement entropy both in gravity and in the conformal dual. In Sec. (3), we present our results for the gravity prediction of the one-loop entanglement entropy due to a higher spin current or a scalar. For a higher spin s current, we provide a closed formula for all the terms starting from x^{2s} and up to x^{4s} . The terms starting from x^{4s} are also computed, but they are only one part of the entanglement entropy that gets corrections also from the so-called 2-CDW contributions which are later. For a scalar field with scaling dimension Δ , we provide similar expansions starting from $x^{2\Delta}$ for generic values of Δ . In Sec. (2), we present a detailed CFT check of the scalar result for the terms up to $\mathcal{O}(x^{2\Delta+5})$. Finally, in Sec. (5), we analyse the 2-CDW contributions providing, among other things, an analytic formula for the leading $\mathcal{O}(x^{4s})$ correction to the entanglement entropy of a higher spin s current.

2 Setup of the gravity and CFT calculations

As we outlined in the Introduction, the ground state Rényi entropy of a quantum system is defined in general terms as follows. We split the spatial domain into two subspaces and build the reduced density matrix ρ by taking the trace over one of the two subspaces. Then, the Rényi entropy, for integer $n > 1$ is defined as

$$S_n = -\frac{1}{n-1} \log \text{Tr}(\rho^n). \quad (2.1)$$

The entanglement entropy is the analytic continuation at $n = 1$

$$S = \lim_{n \rightarrow 1} S_n = -\text{Tr}(\rho \log \rho). \quad (2.2)$$

In a 2d dimensional CFT, the above spatial regions are unions of N disjoint intervals with endpoints $\{z_i\}_{i=1,\dots,2N}$ and the Rényi entropy can be computed on a higher genus Riemann surface Σ by the replica trick, or in terms of correlation functions of $2N$ twist operators at the endpoints.

In the former approach, Σ is an n -sheeted cover of the original spacetime, with branch cuts between the pairs of endpoints of the disjoint intervals [29]. In the AdS/CFT correspondence, the partition function of the 2d CFT on Σ is equal to the partition function of the dual gravity theory on a certain quotient AdS_3/Γ with conformal boundary Σ . Here, Γ is a discrete subgroup of the isometry group $PSL(2, \mathbb{C})$ of AdS_3 that realizes Σ as \mathbb{C}/Γ , i.e. a (replica symmetric) Schottky uniformization of Σ (see [12, 16] for details).

The calculation of S_n on the gravity side amounts to the calculation of the higher genus partition function by means of the bulk gravitational path integral over metrics

asymptotic to Σ with suitable boundary conditions [12]. In Euclidean signature AdS_3 is \mathbb{H}^3 and we look for contributions to the bulk path-integral from locally \mathbb{H}^3 handlebody solutions \mathcal{M} such that on the boundary we reconstruct $\Sigma = \mathbb{C}/\Gamma$. The path integral is evaluated in a semiclassical expansion and, at leading order $\mathcal{O}(c)$ at large central charge, we need the renormalised on-shell bulk action $S_{\text{grav}}(\mathcal{M})$ with

$$S_n = -\frac{1}{n-1} S_{\text{grav}}(\mathcal{M}) + \mathcal{O}(c^0). \quad (2.3)$$

The leading term is completely determined by the details of the Schottky uniformization procedure [11, 12] and we shall not discuss it any more.

Quantum corrections start at $\mathcal{O}(c^0)$ and are sensitive to the microscopic details, *i.e.* the specific form of the CFT. Again, AdS/CFT provides a recipe for the calculation. One has to evaluate the 1-loop functional determinants on handle body geometries for the various bulk fields. The general formula is [19]

$$Z_{\mathbb{H}^3/\Gamma}^{\text{one-loop}} = \prod_{\Phi} \prod_{\gamma \in \mathcal{P}} \left[Z_{\mathbb{H}^3/\mathbb{Z}}^{\Phi}(q_{\gamma}, \bar{q}_{\gamma}) \right]^{1/2}, \quad (2.4)$$

where $\{\Phi\}$ are the bulk fields, q_{γ} are the squared eigenvalues of a certain class \mathcal{P} of elements of the Schottky group Γ , to be specified later in full details. The one-loop contribution to the Rényi entropy is then

$$S_n^{\text{one-loop}} = -\frac{1}{n-1} \log Z_{\mathbb{H}^3/\Gamma}^{\text{one-loop}}. \quad (2.5)$$

The building blocks, on the torus \mathbb{C}/\mathbb{Z} have been computed in [19] for various fields. In particular, for a higher spin gauge field with spin s one has

$$Z_{\mathbb{C}/\mathbb{Z}}^{(s)}(q, \bar{q}) = \prod_{n=s}^{\infty} \frac{1}{|1 - q^n|^2}, \quad (2.6)$$

while for a scalar operator with conformal dimensions $(h, \bar{h}) = (\Delta/2, \Delta/2)$, one has

$$Z_{\mathbb{C}/\mathbb{Z}}^{\text{scalar}}(q, \bar{q}) = \prod_{\ell, \ell'=0}^{\infty} \frac{1}{1 - q^{\ell + \frac{\Delta}{2}} \bar{q}^{\ell' + \frac{\Delta}{2}}}. \quad (2.7)$$

The details for the calculation of the partition function (2.4), *i.e.* the definition of \mathcal{P} and q_{γ} can be found in [16] and will be reviewed later. Remarkably, the eigenvalue q_{γ} is independent on Φ and is universal in the sense that it can be computed in pure gravity.

The CFT side of this story is also straightforward and can be performed by the method of [11, 30, 31] whose notation we follow. We consider the case of two-intervals. The trace of the replica density matrix is a twist fields four point function in the orbifold theory on $(\mathbb{C}/\mathbb{Z})^{\otimes n}$

$$\text{Tr}(\rho^n) = \langle \Phi_+(z_1) \Phi_-(z_2) \Phi_+(z_3) \Phi_-(z_4) \rangle_{(\mathbb{C}/\mathbb{Z})^{\otimes n}}. \quad (2.8)$$

Conformal symmetry is used to fix the positions to $(z_1, z_2, z_3, z_4) = (0, \ell, 1, 1 + \ell)$. The calculation is done in the limit of small (conformal invariant) cross-ratio $x = \frac{z_{12}z_{34}}{z_{13}z_{24}} = \ell^2$.

As is known, the calculation can be organised at small x as a sum over correlators of the $SO(2,1) \times SO(2,1)$ conformal families of the orthogonal quasi primaries \mathcal{O}_K appearing in the fusion rule

$$[\Phi_+] \times [\Phi_-] = [1] + \sum_K d_K [\mathcal{O}_K]. \quad (2.9)$$

Denoting by Δ_K the dimension of \mathcal{O}_K and introducing $\alpha_K = \langle \mathcal{O}_K | \mathcal{O}_K \rangle$, the trace can be written as a sum over global conformal blocks [24]

$$\text{Tr}(\rho^n) = x^{-2\Delta_\Phi} \sum_K \frac{d_K^2}{\alpha_K} x^{\Delta_K} |{}_2F_1(h_K, h_K; 2h_K; x)|^2. \quad (2.10)$$

The first factor depending on the twist fields dimension Δ_Φ gives the classical $\mathcal{O}(c)$ contribution to the trace. The remaining part gives the one-loop contribution to the Rényi entropy

$$S_n^{\text{one-loop}} = -\frac{1}{n-1} \log \left[1 + \sum_{K \neq \mathbb{I}} \frac{d_K^2}{\alpha_K} x^{\Delta_K} |{}_2F_1(h_K, h_K; 2h_K; x)|^2 \right], \quad (2.11)$$

where we have separated out the vacuum exchange contribution. Finally, the OPE coefficients d_K can be computed by the simple formula [30, 31]

$$d_K = \ell^{-\Delta_K} \lim_{z \rightarrow \infty} |z^{2h_K}|^2 \langle \mathcal{O}_K(z, \bar{z}) \rangle_{\mathcal{R}_{n,1}}, \quad (2.12)$$

where $\mathcal{R}_{n,1}$ is the n -cover Riemann surface of the single-interval problem for an interval of length ℓ on the infinite line. The explicit calculation exploits the conformal mapping

$$f(z) = (1 - \ell/z)^{1/n}, \quad (2.13)$$

that maps $\mathcal{R}_{n,1}$ to \mathbb{C} .

As we summarised in the introduction, various checks of AdS/CFT matching have been performed in specific cases including, *i.e.* for various bulk fields [21–23]. In particular, this includes the graviton and a higher spin current with spin 3, and 4.

Here, we want to test AdS/CFT in the calculation of the one-loop Rényi entropy associated with a generic scalar fields. The matching of (2.11) and (2.5) is not obvious and we want to explore which operators are relevant on the CFT side to reproduce the gravity result. In the limit $n \rightarrow 1$ this leads to various simplifications that we exploit to test the matching at high orders. The general structure of the result for a scalar field is

$$S^{(\text{scalar}) \text{ one-loop}} = \sum_{r=1}^{\infty} x^{2r\Delta} \sum_{t=0}^{\infty} S^{(r,t)}(\Delta) x^t. \quad (2.14)$$

The terms with $r = 1$ comes on the gravity side from the so-called CDW elements of the Schottky group. On the CFT side, they are associated with descendants of the scalar field. In principle, these can be obtained by acting on the scalar primary with all modes of the energy momentum tensor. Actually, we shall argue and check that, in the $n \rightarrow 1$ limit, it is enough to consider descendants obtained by acting with L_{-1} and \bar{L}_{-1} alone, that is ∂ ,

$\bar{\partial}$. This remark greatly simplifies the calculation and allowed us to check equality of (2.11) and (2.5) for the terms with t up to 5.

The terms with $r > 1$ are more complicated and will be discussed later. On the gravity side they are associated with r -CDW group elements. On the CFT side, they come from multiple insertions of the scalar operator.

3 The gravity calculation

3.1 Preliminary results

Let us briefly review the results of [16] in a compact way. Let M be the 2×2 diagonal matrix

$$M = \text{diag}(e^{i\pi(1+1/n)}, e^{i\pi(1-1/n)}), \quad (3.1)$$

and T the 2×2 matrix

$$T = \begin{pmatrix} f(y) & f(-y) \\ -f(-y) & -f(y) \end{pmatrix} \quad (3.2)$$

where y is the branch of $x = \frac{4y}{(y+1)^2}$ such that $y = \frac{1}{4}x + \dots$. Remarkably, the matrix T obeys the simple property $y T^2 = \mathbb{I}$. This means that

$$y [f(y)^2 - f(-y)^2] = 1, \quad (3.3)$$

and this relation fixes the even part of T in terms of the odd part. In particular, from the $\mathcal{O}(y^3)$ result of [16], we immediately obtain the $\mathcal{O}(y^4)$ expression of $f(y)$

$$\begin{aligned} f(y) = & \frac{n}{2y} + \frac{1}{2n} + \frac{(-n^4 - n^2 + 2)y}{18n^3} + \frac{(n^4 + n^2 - 2)y^2}{18n^5} \\ & + \frac{(-76n^8 - 80n^6 + 102n^4 + 205n^2 - 151)y^3}{4050n^7} \\ & + \frac{(101n^8 + 130n^6 - 177n^4 - 305n^2 + 251)y^4}{4050n^9} + \mathcal{O}(y^5). \end{aligned} \quad (3.4)$$

The Schottky generators are the matrices

$$\mathcal{L}_k = M^k T^{-1} M T M^{-(k-1)}, \quad k = 1, \dots, n-1. \quad (3.5)$$

At leading order $r = 1$ in (2.14), we need to consider the so-called consecutively decreasing words (CDW) of the form

$$\gamma_{k,m} = \mathcal{L}_{k+m} \mathcal{L}_{k+m-1} \cdots \mathcal{L}_{m+1} = M^{m+k} T^{-1} M^k T M^{-m}. \quad (3.6)$$

The quantity q_γ is defined as the root of the secular equation

$$\det(\gamma_{k,m} - q_\gamma^{-1/2} \mathbb{I}_{2 \times 2}) = 0, \quad (3.7)$$

with $|q_\gamma| < 1$. Clearly, it does not depend on m , but is a function of k . Summing over the CDW elements, we have to remind that at each k we have $n - k$ inequivalent words, one

for each m , plus their inverses. Solving perturbatively (3.7) and expanding in powers of x , we find

$$\begin{aligned}
q_\gamma = & \frac{\kappa^4 x^2}{16n^4} + x^3 \left(\frac{\kappa^6}{16n^6} + \frac{\kappa^4 (n^2 - 1)}{16n^6} \right) + x^4 \left(\frac{7\kappa^8}{128n^8} + \frac{3\kappa^6 (n^2 - 1)}{32n^8} + \frac{\kappa^4 (n^2 - 1) (65n^2 - 41)}{1152n^8} \right) \\
& + x^5 \left(\frac{3\kappa^{10}}{64n^{10}} + \frac{7\kappa^8 (n^2 - 1)}{64n^{10}} + \frac{\kappa^6 (n^2 - 1) (83n^2 - 59)}{768n^{10}} + \frac{\kappa^4 (n^2 - 1) (116n^4 - 125n^2 + 33)}{2304n^{10}} \right) \\
& + x^6 \left(\frac{165\kappa^{12}}{4096n^{12}} + \frac{15\kappa^{10} (n^2 - 1)}{128n^{12}} + \frac{7\kappa^8 (n^2 - 1) (101n^2 - 77)}{4608n^{12}} + \frac{\kappa^6 (n^2 - 1) (7n^2 - 4) (25n^2 - 17)}{1536n^{12}} \right. \\
& \left. + \frac{\kappa^4 (n^2 - 1) (374279n^6 - 530301n^4 + 244341n^2 - 36479)}{8294400n^{12}} \right) \\
& + x^7 \left(\frac{143\kappa^{14}}{4096n^{14}} + \frac{495\kappa^{12} (n^2 - 1)}{4096n^{14}} + \frac{5\kappa^{10} (n^2 - 1) (119n^2 - 95)}{3072n^{14}} \right. \\
& + \frac{7\kappa^8 (n^2 - 1) (246n^4 - 337n^2 + 115)}{9216n^{14}} + \frac{\kappa^6 (n^2 - 1) (13298n^6 - 22362n^4 + 12417n^2 - 2273)}{115200n^{14}} \\
& \left. + \frac{\kappa^4 (n^2 - 1) (112679n^8 - 190053n^6 + 116829n^4 - 30887n^2 + 2952)}{2764800n^{14}} \right) + \mathcal{O}(x^8), \tag{3.8}
\end{aligned}$$

where we have defined

$$\kappa = \frac{1}{\sin\left(\frac{k\pi}{n}\right)}. \tag{3.9}$$

3.2 The entanglement limit $n = 1$

If we want to present results in the limit $n \rightarrow 1$, then several remarks simplify the calculation. The quantity q_γ involves even powers of κ and is thus invariant under $k \rightarrow n - k$. The contribution to the partition function of the various bulk fields can be expanded in powers of x , and the final summation over k will always involve elementary sums of the form (the factor $2(n - k)$ counts the CDW)

$$f_n(p) = \sum_{k=1}^{n-1} 2(n - k) \frac{1}{\sin^{2p}\left(\frac{k\pi}{n}\right)} = n \sum_{k=1}^{n-1} \frac{1}{\sin^{2p}\left(\frac{k\pi}{n}\right)} \tag{3.10}$$

The analytic continuation of this sum has been discussed in [16, 31] and one finds the finite result

$$\lim_{n \rightarrow 1} \frac{1}{n-1} f_n(p) = \frac{\sqrt{\pi} \Gamma(p+1)}{2 \Gamma\left(p + \frac{3}{2}\right)}. \tag{3.11}$$

The fact that the singularity as $n \rightarrow 1$ is fully absorbed by the above sum has an important consequence. The $n \rightarrow 1$ limit of the gravity partition function can be computed by setting from the scratch $n = 1$ in the various coefficients of q_γ keeping the dependence on n only inside κ . Thus, we can work with the much simpler expression

$$q_\gamma = \frac{\kappa^4 x^2}{16} + \frac{\kappa^6 x^3}{16} + \frac{7\kappa^8 x^4}{128} + \frac{3\kappa^{10} x^5}{64} + \frac{165\kappa^{12} x^6}{4096} + \frac{143\kappa^{14} x^7}{4096} + \mathcal{O}(x^8). \tag{3.12}$$

It is very tempting to resum this expression in the following closed form, at least perturbatively in x ,

$$q_\gamma = \frac{4\sqrt{1-\kappa^2x}(\kappa^2x-2)}{\kappa^4x^2} - \frac{8(\kappa^2x-1)}{\kappa^4x^2} + 1, \quad (3.13)$$

that means

$$q_\gamma = 4 \sum_{p=2}^{\infty} \left[\binom{1/2}{p+1} + 2 \binom{1/2}{p+2} \right] (-1)^{p+1} (\kappa^2x)^p. \quad (3.14)$$

This is an all order result for the CDW quantity q_γ that can be used to obtain various explicit expansions.

3.3 Results in the higher spin current case

The contribution we want to compute is that of a higher spin s current

$$S^{(s)\text{ one-loop}} = -\lim_{n \rightarrow 1} \frac{1}{n-1} \log \prod_{\gamma \in \mathcal{P}} \prod_{p=s}^{\infty} \frac{1}{|1-q_\gamma^p|}. \quad (3.15)$$

We consider the CDW contribution that starts at x^{2s} with corrections in powers of x . It is obtained by considering the subset $\mathcal{P}_1 \subset \mathcal{P}$ of Schottky group elements that are CDW. For these elements, we have determined q_γ in, say, (3.14). We expand the logarithm and find

$$\begin{aligned} S_{\text{CDW}}^{(s)\text{ one-loop}} &= -\lim_{n \rightarrow 1} \frac{1}{n-1} \sum_{\gamma \in \mathcal{P}_1} \sum_{p=s}^{\infty} \left(q_\gamma^p + \frac{1}{2} q_\gamma^{2p} + \dots \right) = \\ &= -\lim_{n \rightarrow 1} \frac{1}{n-1} \sum_{\gamma \in \mathcal{P}_1} \left(\underbrace{\frac{q_\gamma^s}{1-q_\gamma}}_{(I)} + \underbrace{\frac{1}{2} \frac{q_\gamma^{2s}}{1-q_\gamma^2}}_{(II)} + \dots \right) = S_{\text{CDW},(I)}^{(s)\text{ one-loop}} + S_{\text{CDW},(II)}^{(s)\text{ one-loop}} + \dots, \end{aligned} \quad (3.16)$$

where we took into account the fact that q_γ is real. The first term (I) in the sum takes the form of x^{2s} times a power series in x . Its terms up to $\mathcal{O}(x^{4s})$ are the exact contributions to the Rényi entropy. The next terms must be combined with the second term (II) in the sum which takes the form of x^{4s} times a power series in x . This part of the CDW contribution gets corrections from the Rényi entropy due to the 2-CDW Schottky group elements that also starts at order $\mathcal{O}(x^{4s})$, see later sections. Expanding q_γ using (3.13) and performing the limit with (3.11), we find the expansion

$$\begin{aligned} S_{\text{CDW},(I)}^{(s)\text{ one-loop}} &= -\left(\frac{x}{4}\right)^{2s} \frac{\sqrt{\pi}}{2} \frac{\Gamma(2s+1)}{\Gamma(2s+\frac{3}{2})} \left[1 + \frac{2s(2s+1)x}{4s+3} + \frac{(s+1)(2s+1)^2(4s+1)x^2}{2(4s+3)(4s+5)} \right. \\ &+ \frac{(s+1)^2(2s+1)^2(2s+3)x^3}{3(4s+5)(4s+7)} + \frac{(s+1)^2(s+2)(2s+1)(2s+3)^2x^4}{12(4s+7)(4s+9)} \\ &\left. + \frac{(s+1)^2(s+2)^2(2s+1)(2s+3)^2(2s+5)x^5}{30(4s+3)(4s+9)(4s+11)} + \mathcal{O}(x^6) \right], \end{aligned} \quad (3.17)$$

and

$$S_{\text{CDW},(II)}^{(s)\text{ one-loop}} = -\left(\frac{x}{4}\right)^{4s} \frac{\sqrt{\pi}}{4} \frac{\Gamma(4s+1)}{\Gamma(4s+\frac{3}{2})} \left[1 + \frac{4s(4s+1)x}{8s+3} + \frac{2s(2s+1)(4s+1)x^2}{8s+5} \right]$$

$$\begin{aligned}
& + \frac{4s(2s+1)^2(4s+1)(4s+3)x^3}{3(8s+3)(8s+7)} + \frac{(s+1)(2s+1)^2(4s+1)(4s+3)(256s^3+448s^2+204s+3)x^4}{6(8s+3)(8s+5)(8s+7)(8s+9)} \\
& + \frac{2(s+1)^2(2s+1)^2(4s+1)(4s+3)(4s+5)(256s^3+576s^2+348s+15)x^5}{15(8s+3)(8s+5)(8s+7)(8s+9)(8s+11)} + \mathcal{O}(x^6) \Big]. \quad (3.18)
\end{aligned}$$

Of course, there is no difficulty in computing the analogous higher order terms $S_{\text{CDW,(III)}}^{(s) \text{ one-loop}}$ and so on. The expansion (3.17) can be extended without difficulty, given the closed formula for q_γ . Doing so, one notices a remarkable regularity in its coefficients. After some work, we found the following very nice compact formula

$$S_{\text{CDW,(I)}}^{(s) \text{ one-loop}} = -\frac{\sqrt{\pi}}{2} \left(\frac{x}{4}\right)^{2s} \sum_{p=0}^{\infty} \frac{\Gamma(p+2s+1)\Gamma(2p+4s-1)}{\Gamma(p+1)\Gamma(p+2s+\frac{3}{2})\Gamma(p+4s-1)} \left(\frac{x}{4}\right)^p. \quad (3.19)$$

It can be summed in closed form with the very nice result

$$S_{\text{CDW,(I)}}^{(s) \text{ one-loop}} = -\frac{\sqrt{\pi}}{2} \left(\frac{x}{4}\right)^{2s} \frac{\Gamma(2s+1)}{\Gamma(2s+\frac{3}{2})} {}_3F_2\left(2s, 2s-\frac{1}{2}, 2s+1; 2s+\frac{3}{2}, 4s-1; x\right), \quad (3.20)$$

that we anticipated in the Introduction.

3.3.1 Comparison with known results

For generic s , the first two terms of (3.17) agree with the general result of [24]. For the graviton, $s = 2$, we obtain

$$\begin{aligned}
S_{\text{CDW,(I)}}^{(2) \text{ one-loop}} &= -\frac{x^4}{630} - \frac{2x^5}{693} - \frac{15x^6}{4004} - \frac{x^7}{234} - \frac{7x^8}{1530} - \frac{84x^9}{17765} - \frac{x^{10}}{209} - \frac{10x^{11}}{2093} + \mathcal{O}(x^{12}), \\
S_{\text{CDW,(II)}}^{(2) \text{ one-loop}} &= -\frac{x^8}{437580} - \frac{2x^9}{230945} - \frac{x^{10}}{51051} - \frac{10x^{11}}{289731} + \mathcal{O}(x^{12}), \\
S_{\text{CDW,(III)}}^{(2) \text{ one-loop}} &= \mathcal{O}(x^{12}),
\end{aligned} \quad (3.21)$$

and therefore the $\mathcal{O}(x^{12})$ CDW one-loop entropy is

$$\begin{aligned}
S_{\text{CDW}}^{(2) \text{ one-loop}} &= -\frac{x^4}{630} - \frac{2x^5}{693} - \frac{15x^6}{4004} - \frac{x^7}{234} - \frac{2003x^8}{437580} - \frac{1094x^9}{230945} \\
&\quad - \frac{4660x^{10}}{969969} - \frac{9760x^{11}}{2028117} + \mathcal{O}(x^{12}), \quad (3.22)
\end{aligned}$$

in full agreement with Eq. (63) of [16]. For the spin $s = 3$ current, we obtain

$$S_{\text{CDW}}^{(3) \text{ one-loop}} = -\frac{x^6}{12012} - \frac{x^7}{4290} - \frac{7x^8}{16830} - \frac{28x^9}{46189} - \frac{15x^{10}}{19019} - \frac{2x^{11}}{2093} + \mathcal{O}(x^{12}), \quad (3.23)$$

and the first three terms are in agreement with the $n \rightarrow 1$ limit of the $\mathcal{O}(x^8)$ calculation (2.8) in [22]. Finally, for the spin $s = 4$ current, we obtain

$$S_{\text{CDW}}^{(4) \text{ one-loop}} = -\frac{x^8}{218790} - \frac{4x^9}{230945} - \frac{3x^{10}}{76076} - \frac{5x^{11}}{71162} + \mathcal{O}(x^{12}), \quad (3.24)$$

and the first term is in agreement with the entanglement $n \rightarrow 1$ limit of the $\mathcal{O}(x^8)$ calculation (2.9) in [22].

3.4 Results in the scalar field case

In this case, the contribution we want to compute is that associated with (2.7)

$$S^{(\text{scalar}) \text{ one-loop}} = -\frac{1}{2} \lim_{n \rightarrow 1} \frac{1}{n-1} \log \prod_{\gamma \in \mathcal{P}} \prod_{\ell, \ell'=0}^{\infty} \frac{1}{1 - q_{\gamma}^{\ell + \frac{\Delta}{2}} \bar{q}_{\gamma}^{\ell' + \frac{\Delta}{2}}}. \quad (3.25)$$

We can repeat the analysis of the previous section and write

$$\begin{aligned} S_{\text{CDW}}^{(s) \text{ one-loop}} &= -\frac{1}{2} \lim_{n \rightarrow 1} \frac{1}{n-1} \sum_{\gamma \in \mathcal{P}_1} \sum_{p=0}^{\infty} (p+1) \left(q_{\gamma}^{\Delta+p} + \frac{1}{2} q_{\gamma}^{2(\Delta+p)} + \dots \right) \\ &= -\frac{1}{2} \lim_{n \rightarrow 1} \frac{1}{n-1} \sum_{\gamma \in \mathcal{P}_1} \left(\underbrace{\frac{q_{\gamma}^{\Delta}}{(q_{\gamma} - 1)^2}}_{(I)} + \underbrace{\frac{1}{2} \frac{q_{\gamma}^{2\Delta}}{(q_{\gamma}^2 - 1)^2}}_{(II)} + \dots \right) = \\ &= S_{\text{CDW},(I)}^{(\text{scalar}) \text{ one-loop}} + S_{\text{CDW},(I)}^{(\text{scalar}) \text{ one-loop}} + \dots \end{aligned} \quad (3.26)$$

Expanding q_{γ} using (3.13) and performing the limit with (3.11), we find the expansions

$$\begin{aligned} S_{\text{CDW},(I)}^{(\text{scalar}) \text{ one-loop}} &= -\left(\frac{x}{4}\right)^{2\Delta} \frac{\sqrt{\pi}}{4} \frac{\Gamma(2\Delta+1)}{\Gamma(2\Delta+\frac{3}{2})} \left[\right. \\ &1 + \frac{2\Delta(2\Delta+1)x}{4\Delta+3} + \frac{(\Delta+1)(2\Delta+1)(4\Delta^2+3\Delta+1)x^2}{(4\Delta+3)(4\Delta+5)} \\ &+ \frac{2(\Delta+1)^2(2\Delta+1)(2\Delta+3)(4\Delta^2+5\Delta+3)x^3}{3(4\Delta+3)(4\Delta+5)(4\Delta+7)} \\ &+ \frac{(\Delta+1)(\Delta+2)(2\Delta+1)(2\Delta+3)(32\Delta^4+144\Delta^3+262\Delta^2+237\Delta+93)x^4}{12(4\Delta+3)(4\Delta+5)(4\Delta+7)(4\Delta+9)} \\ &\left. + \frac{(\Delta+1)(\Delta+2)^2(2\Delta+1)(2\Delta+3)(2\Delta+5)(32\Delta^4+176\Delta^3+398\Delta^2+449\Delta+225)x^5}{30(4\Delta+3)(4\Delta+5)(4\Delta+7)(4\Delta+9)(4\Delta+11)} + \mathcal{O}(x^6) \right], \end{aligned}$$

and

$$\begin{aligned} S_{\text{CDW},(II)}^{(\text{scalar}) \text{ one-loop}} &= -\left(\frac{x}{4}\right)^{4\Delta} \frac{\sqrt{\pi}}{8} \frac{\Gamma(4\Delta+1)}{\Gamma(4\Delta+\frac{3}{2})} \left[\right. \\ &1 + \frac{4\Delta(4\Delta+1)x}{8\Delta+3} + \frac{2\Delta(2\Delta+1)(4\Delta+1)x^2}{8\Delta+5} + \frac{4\Delta(2\Delta+1)^2(4\Delta+1)(4\Delta+3)x^3}{3(8\Delta+3)(8\Delta+7)} \\ &+ \frac{(\Delta+1)(2\Delta+1)(4\Delta+1)(4\Delta+3)(256\Delta^4+576\Delta^3+428\Delta^2+105\Delta+3)x^4}{3(8\Delta+3)(8\Delta+5)(8\Delta+7)(8\Delta+9)} \\ &+ \frac{4(\Delta+1)^2(2\Delta+1)(4\Delta+1)(4\Delta+3)(4\Delta+5)(8\Delta+1)(32\Delta^3+84\Delta^2+69\Delta+15)x^5}{15(8\Delta+3)(8\Delta+5)(8\Delta+7)(8\Delta+9)(8\Delta+11)} \\ &\left. + \mathcal{O}(x^6) \right]. \end{aligned} \quad (3.27)$$

The leading term is one half of the analogous result for the higher spin currents. This is due to the fact that the scalar partition function starts with the single state $|\frac{\Delta}{2}, \frac{\Delta}{2}\rangle$, while

the higher spin current provides two low-lying states $J_{-s}|0\rangle$ and $\bar{J}_{-s}|0\rangle$. In other words, the difference is in the value of the multiplicity \mathcal{N} appearing in (41) of [16]. The bulk scalar field has been studied in the one-interval case in [20]. No results are available in the two-interval case.

3.5 Exponentiation in the classical limit

It is interesting to take the limit of (3.17), (3.18), (3.27), and (3.27) when $\Delta \rightarrow \infty$ with fixed product $\xi = s x$ (higher spin current) or $\xi = \Delta x$ (scalar). In the three cases, the square brackets takes the form of a simple exponential

$$\left[\dots \right]_{(3.17)} \rightarrow e^{\xi}, \quad \left[\dots \right]_{(3.18)} \rightarrow e^{2\xi}, \quad \left[\dots \right]_{(3.27)} \rightarrow e^{\xi}, \quad \left[\dots \right]_{(3.27)} \rightarrow e^{2\xi}. \quad (3.28)$$

It would be very interesting to understand the relation between this exponentiation and the analogous properties of the semiclassical conformal blocks [32].

4 CFT check of the scalar field entanglement entropy

It is interesting to reproduce the result (3.27) from a direct CFT calculation according to (2.11). To this aim, we consider a primary field $X(z, \bar{z})$ with conformal weights $(L_0, \bar{L}_0) = (h, \bar{h})$. In the end, we shall specialise to $h = \bar{h} = \frac{\Delta}{2}$.

The recipe (2.11) requires to list all quasi-primaries belonging to the replica space $(\mathbb{C}/\mathbb{Z})^{\otimes n}$. If we are interested in reproducing (3.27), then we can restrict ourselves to linear combinations of operators of the form

$$\mathcal{O}(z, \bar{z}) = \mathcal{A}(z_{j_1}, \bar{z}_{j_1}) \otimes \mathcal{B}(z_{j_2}, \bar{z}_{j_2}), \quad (4.1)$$

where z_{j_p} is the image of z on sheet j_p , and $j_1 \neq j_2$. Also, \mathcal{A} and \mathcal{B} have to be linear in X . This means that \mathcal{A}, \mathcal{B} are obtained acting on X with the negative modes L_{-n} of the energy momentum tensor T (and its antiholomorphic counterpart). In principle, there are non trivial cases involving L_{-n} with $n > 1$. For instance, the operator $\mathcal{O} = \mathcal{A} \otimes X$ with ¹

$$\mathcal{A} = (TX) - \frac{3}{2(2h+1)} \partial^2 X, \quad (4.2)$$

is quasi-primary. Nevertheless, in the $n \rightarrow 1$ limit, one can exploit a major simplification and consider only operators \mathcal{A}, \mathcal{B} that are obtained from X by applying a certain number of derivatives $\partial, \bar{\partial}$, forgetting at all T .

The reason behind this expectation is the following. The classical $\mathcal{O}(c)$ contribution to the entanglement entropy is expected to be universal. It is determined by the central charge only and fully encoded in the vacuum Verma module. This has been checked in several examples [21–23]. Any operator involving T has a norm α_K growing with c . The associated coefficient d_K can be expanded in inverse powers of the central charge c and the leading term for any operator involving X has to be accompanied by an explicit

¹The operator (TX) is the conformal normal ordered product.

factor $(n^2 - 1)$. This has the consequence that the one-loop $\mathcal{O}(c^0)$ contributions from such operators has the same factor making it vanish at $n = 1$, since the sum over $j_1 \neq j_2$ provides always another factor $n - 1$ cancelling the $1/(n - 1)$ in the definition of the Rényi entropy.

From the gravity point of view, the determinant formula (2.7) counts precisely the descendants of X obtained by acting with the Virasoro operators L_{-1}, \bar{L}_{-1} . The contributions from higher modes $L_{-n}, n > 1$ is encoded in the one-loop partition function of the graviton bulk field, which is (2.6) with $s = 2$.

In the following part of this section, we shall construct all quasi-primaries of the above form and compute for all of them the normalisation α_K and the one-point coefficient d_K .

4.1 Relevant quasi-primaries up to level 5

We define the level as $L_0 + \bar{L}_0 - 2\Delta$, with $\Delta = h + \bar{h}$ being the scaling dimension of X . At level 0, there is a single field

$$\mathcal{O}^{(0,1)} = XX, \quad (4.3)$$

$$\alpha^{(0,1)} = \alpha_X^2, \quad (4.4)$$

$$d^{(0,1)} = \frac{1}{(2n s_{j_1 j_2})^{2\Delta}} \alpha_X, \quad (4.5)$$

where $s_{j_1 j_2} = \sin\left(\pi \frac{j_1 - j_2}{n}\right)$. The notation is of course $XX \equiv X_{j_1} \otimes X_{j_2}$, and α_X^2 is the normalisation of the 2-point function $\langle XX \rangle$.

At level 1, we find the two conjugate operators

$$-i \mathcal{O}^{(1,1)} = \partial X X - X \partial X, \quad (4.6)$$

$$\alpha^{(1,1)} = 4h \alpha_X^2, \quad (4.7)$$

$$d^{(1,1)} = 4h \frac{c_{j_1 j_2}}{(2n s_{j_1 j_2})^{2\Delta+1}} \alpha_X, \quad (4.8)$$

$$-i \mathcal{O}^{(1,2)} = \bar{\partial} X X - X \bar{\partial} X, \quad (4.9)$$

$$\alpha^{(1,2)} = 4\bar{h} \alpha_X^2, \quad (4.10)$$

$$d^{(1,2)} = 4\bar{h} \frac{c_{j_1 j_2}}{(2n s_{j_1 j_2})^{2\Delta+1}} \alpha_X, \quad (4.11)$$

where $c_{j_1 j_2} = \cos\left(\pi \frac{j_1 - j_2}{n}\right)$. In the following, it will be convenient to write $\bar{\mathcal{O}}$ to denote the operators that is obtained from \mathcal{O} by swapping $\partial \leftrightarrow \bar{\partial}$. The coefficients α_K (as well as d_K) of the two operators are related by the swap $h \leftrightarrow \bar{h}$. Such pairs of conjugate operators give the same contribution to (3.27) which is evaluated at $h = \bar{h}$.

At level 2, the quasi-primaries are

$$\mathcal{O}^{(2,1)} = \partial X \partial X - \frac{h}{2h+1} (\partial^2 X X + X \partial^2 X), \quad (4.12)$$

$$\alpha^{(2,1)} = \frac{4h^2(4h+1)}{2h+1} \alpha_X^2, \quad (4.13)$$

$$d^{(2,1)} = 2h \frac{4h+1-4hs_{j_1j_2}^2}{(2ns_{j_1j_2})^{2\Delta+2}} \alpha_X, \quad (4.14)$$

$$\mathcal{O}^{(2,2)} = \overline{\mathcal{O}^{(2,1)}}, \quad (4.15)$$

$$\mathcal{O}^{(2,3)} = X\partial\bar{\partial}X + \partial\bar{\partial}XX - \partial X\bar{\partial}X - \bar{\partial}X\partial X, \quad (4.16)$$

$$\alpha^{(2,3)} = 16h\bar{h}\alpha_X^2, \quad (4.17)$$

$$d^{(2,3)} = 16h\bar{h} \frac{c_{j_1j_2}^2}{(2ns_{j_1j_2})^{2\Delta+2}} \alpha_X. \quad (4.18)$$

These operators are a subset of the operators considered in section (3.3) of [23]. As we mentioned before, we can skip any operator involving the energy-momentum tensor in this calculation. The next operators at levels 3, 4, 5 are

Level 3

$$-i\mathcal{O}^{(3,1)} = \partial X\partial\bar{\partial}X - \partial\bar{\partial}X\partial X + \frac{h}{2h+1}(\partial^2\bar{\partial}XX - X\partial^2\bar{\partial}X + \bar{\partial}X\partial^2X - \partial^2X\bar{\partial}X), \quad (4.19)$$

$$\alpha^{(3,1)} = \frac{16h^2\bar{h}(4h+1)}{2h+1} \alpha_X^2, \quad (4.20)$$

$$d^{(3,1)} = 8h\bar{h} \frac{c_{j_1j_2}(4h+1-4hs_{j_1j_2}^2)}{(2ns_{j_1j_2})^{2\Delta+3}} \alpha_X, \quad (4.21)$$

$$\mathcal{O}^{(3,2)} = \overline{\mathcal{O}^{(3,1)}}, \quad (4.22)$$

$$-i\mathcal{O}^{(3,3)} = \partial^2X\partial X - \partial X\partial^2X + \frac{h}{3(h+1)}(X\partial^3X - \partial^3XX), \quad (4.23)$$

$$\alpha^{(3,3)} = \frac{16h^2(2h+1)(4h+3)}{3(h+1)} \alpha_X^2, \quad (4.24)$$

$$d^{(3,3)} = \frac{8}{3}h(2h+1) \frac{c_{j_1j_2}(4h+3-4hs_{j_1j_2}^2)}{(2ns_{j_1j_2})^{2\Delta+3}} \alpha_X, \quad (4.25)$$

$$\mathcal{O}^{(3,4)} = \overline{\mathcal{O}^{(3,3)}}. \quad (4.26)$$

Level 4

$$\mathcal{O}^{(4,1)} = \partial^3X\partial X + \partial X\partial^3X - \frac{h}{2(2h+3)}(\partial^4XX + X\partial^4X) - \frac{3(h+1)}{2h+1}\partial^2X\partial^2X, \quad (4.27)$$

$$\alpha^{(4,1)} = \frac{48h^2(h+1)(4h+3)(4h+5)}{2h+3}, \quad (4.28)$$

$$d^{(4,1)} = 4h(h+1) \frac{16h^2c_{j_1j_2}^4 + 8h(s_{j_1j_2}^4 - 5s_{j_1j_2}^2 + 4) - 12s_{j_1j_2}^2 + 15}{(2ns_{j_1j_2})^{2\Delta+4}} \alpha_X, \quad (4.29)$$

$$\mathcal{O}^{(4,2)} = \overline{\mathcal{O}^{(4,1)}}, \quad (4.30)$$

$$\begin{aligned}\mathcal{O}^{(4,3)} = & \partial^2 X \partial \bar{\partial} X + \partial \bar{\partial} X \partial^2 X - \partial X \partial^2 \bar{\partial} X - \partial^2 \bar{\partial} X \partial X \\ & + \frac{h}{3(h+1)} (\partial^3 \bar{\partial} X X - \partial^3 X \bar{\partial} X - \bar{\partial} X \partial^3 X + X \partial^3 \bar{\partial} X),\end{aligned}\quad (4.31)$$

$$\alpha^{(4,3)} = \frac{64h^2\bar{h}(2h+1)(4h+3)}{3(h+1)}, \quad (4.32)$$

$$d^{(4,3)} = \frac{32}{3} h \bar{h} (2h+1) \frac{c_{j_1 j_2}^2 (4h c_{j_1 j_2}^2 + 3)}{(2n s_{j_1 j_2})^{2\Delta+4}} \alpha_X, \quad (4.33)$$

$$\mathcal{O}^{(4,4)} = \overline{\mathcal{O}^{(4,3)}}, \quad (4.34)$$

$$\begin{aligned}\mathcal{O}^{(4,5)} = & \partial \bar{\partial} X \partial \bar{\partial} X - \frac{h}{2h+1} (\partial^2 \bar{\partial} X \bar{\partial} X + \bar{\partial} X \partial^2 \bar{\partial} X) - \frac{\bar{h}}{2\bar{h}+1} (\bar{\partial}^2 \partial X \partial X + \partial X \bar{\partial}^2 \partial X) \\ & + \frac{h\bar{h}}{(2h+1)(2\bar{h}+1)} (\partial^2 \bar{\partial}^2 X X + \partial^2 X \bar{\partial}^2 X + \bar{\partial}^2 X \partial^2 X + X \partial^2 \bar{\partial}^2 X),\end{aligned}\quad (4.35)$$

$$\alpha^{(4,5)} = \frac{16h^2\bar{h}^2(4h+1)(4\bar{h}+1)}{(2h+1)(2\bar{h}+1)}, \quad (4.36)$$

$$d^{(4,5)} = 4h\bar{h} \frac{(4hc_{j_1 j_2}^2 + 1)(4\bar{h}c_{j_1 j_2}^2 + 1)}{(2n s_{j_1 j_2})^{2\Delta+4}} \alpha_X. \quad (4.37)$$

Level 5

Let us define $\mathfrak{s} = s_{j_1 j_2}$, $\mathfrak{c} = c_{j_1 j_2}$.

$$\begin{aligned}-i\mathcal{O}^{(5,1)} = & \partial^2 X \partial^3 X - \partial^3 X \partial^2 X + \frac{h(2h+1)}{10(h+2)(2h+3)} (X \partial^5 X - \partial^5 X X) \\ & + \frac{2h+1}{2(2h+3)} (\partial^4 X \partial X - \partial X \partial^4 X),\end{aligned}\quad (4.38)$$

$$\alpha^{(5,1)} = \frac{192h^2(h+1)(2h+1)^2(4h+5)(4h+7)}{5(h+2)(2h+3)}, \quad (4.39)$$

$$d^{(5,1)} = \frac{16}{5} h(h+1)(2h+1) \frac{\mathfrak{c} (16h^2\mathfrak{c}^4 + 8h(\mathfrak{s}^4 - 7\mathfrak{s}^2 + 6) - 20\mathfrak{s}^2 + 35)}{(2n s_{j_1 j_2})^{2\Delta+5}} \alpha_X, \quad (4.40)$$

$$\mathcal{O}^{(5,2)} = \overline{\mathcal{O}^{(5,1)}}, \quad (4.41)$$

$$\begin{aligned}-i\mathcal{O}^{(5,3)} = & \partial^2 \bar{\partial} X \partial^2 X - \partial^2 X \partial^2 \bar{\partial} X \\ & + \frac{h(2h+1)}{6(h+1)(2h+3)} (\partial^4 \bar{\partial} X X - \partial^4 X \bar{\partial} X + \bar{\partial} X \partial^4 X - X \partial^4 \bar{\partial} X) \\ & + \frac{2h+1}{3(h+1)} (\partial^3 X \partial \bar{\partial} X - \partial^3 \bar{\partial} X \partial X - \partial \bar{\partial} X \partial^3 X + \partial X \partial^3 \bar{\partial} X),\end{aligned}\quad (4.42)$$

$$\alpha^{(5,3)} = \frac{64h^2\bar{h}(2h+1)^2(4h+3)(4h+5)}{3(h+1)(2h+3)}, \quad (4.43)$$

$$d^{(5,3)} = \frac{16}{3} h \bar{h} (2h+1) \frac{\mathfrak{c} (16h^2\mathfrak{c}^4 + 8h(\mathfrak{s}^4 - 5\mathfrak{s}^2 + 4) - 12\mathfrak{s}^2 + 15)}{(2n s_{j_1 j_2})^{2\Delta+5}} \alpha_X, \quad (4.44)$$

$$\mathcal{O}^{(5,4)} = \overline{\mathcal{O}^{(5,3)}}, \quad (4.45)$$

$$\begin{aligned} -i\mathcal{O}^{(5,5)} &= \partial^2 \bar{\partial} X \partial \bar{\partial} X - \partial \bar{\partial} X \partial^2 \bar{\partial} X + \frac{h}{3(h+1)} (\bar{\partial} X \partial^3 \bar{\partial} X - \partial^3 \bar{\partial} X \bar{\partial} X) \\ &\quad + \frac{\bar{h}}{2\bar{h}+1} (\partial X \partial^2 \bar{\partial}^2 X + \partial \bar{\partial}^2 X \partial^2 X - \partial^2 X \partial \bar{\partial}^2 X - \partial^2 \bar{\partial}^2 X \partial X) \\ &\quad + \frac{h\bar{h}}{3(h+1)(2\bar{h}+1)} (\partial^3 \bar{\partial}^2 X X + \partial^3 X \bar{\partial}^2 X - \bar{\partial}^2 X \partial^3 X - X \partial^3 \bar{\partial}^2 X), \end{aligned} \quad (4.46)$$

$$a^{(5,5)} = \frac{64h^2\bar{h}^2(2h+1)(4h+3)(4\bar{h}+1)}{3(h+1)(2\bar{h}+1)}, \quad (4.47)$$

$$d^{(5,5)} = \frac{16}{3} h \bar{h} (2h+1) \frac{c(4hc^2+3)(4\bar{h}c^2+1)}{(2n s_{j_1 j_2})^{2\Delta+5}} \alpha_X, \quad (4.48)$$

$$\mathcal{O}^{(5,6)} = \overline{\mathcal{O}^{(5,5)}}. \quad (4.49)$$

4.2 Matching the gravity side

We now plug the results collected in the last section into (2.11) and expand the hypergeometric conformal blocks in order to recover the full $\mathcal{O}(x^{2\Delta+5})$ gravity result (3.27) after going to the $n \rightarrow 1$ limit. Notice that the involved sums are the same as in the gravity calculation since the sum over $j_1 < j_2$ gives back the sum over $k = |j_2 - j_1|$ with multiplicity $n - k$. The first two terms reproduce the calculation discussed in [24]. Consider now, as an example, the next term which is at $\mathcal{O}(x^{2\Delta+2})$. The relevant terms in (2.11) are

$$\begin{aligned} & -\frac{1}{2(2n)^{4\Delta}} \left\{ \frac{1}{s_k^{4\Delta}} \left[1 + 2hx + \frac{h(8h^2+5h+1)x^2}{4h+1} + \dots \right] \right. \\ & + 2\Delta \frac{1}{(2n)^2} \frac{c_k^2}{s_k^{4\Delta+2}} \left[2 + (4h+1)x + \dots \right] x \\ & \left. + x^2 \left(\frac{16}{(2n)^4} h^2 \frac{c_k^4}{s_k^{4\Delta+4}} + \frac{2}{(2n)^4} \frac{2h+1}{4h+1} \frac{(4h+1-4hs_k^2)^2}{s_k^{4\Delta+4}} \right) x^2 + \dots \right\} \end{aligned} \quad (4.50)$$

where $s_k = \sin(\pi k/n)$, $c_k = \cos(\pi k/n)$, $h = \Delta/2$, and the square brackets are the expansions of the conformal blocks. Going to the $n \rightarrow 1$ limit we reduce the coefficient of x^2 in this expression to the simple quantity

$$-4^{-2\Delta-2} (4\Delta^2 + 3\Delta + 1) \frac{1}{s_k^{4\Delta+4}} \xrightarrow{(3.11)} -\frac{\sqrt{\pi} 2^{-4\Delta-5} (4\Delta^2 + 3\Delta + 1) \Gamma(2\Delta + 3)}{\Gamma(2\Delta + \frac{7}{2})}, \quad (4.51)$$

where the r.h.s. is the result of the sum over k by means of (3.11). This is in full agreement with the $\mathcal{O}(x^{2\Delta+2})$ in (3.27). A similar check can be repeated for the other terms of (3.27) with full agreement.

5 The 2-CDW contributions

In this section, we discuss the 2-CDW contributions providing an exact formula for the leading $\mathcal{O}(x^{4s})$ term and the tools needed to evaluate the next two terms $\mathcal{O}(x^{4s+1})$ and $\mathcal{O}(x^{4s+2})$. In particular, we shall combine these results with the analysis of the previous sections in order to compute the exact entanglement entropy for a spin $s = 2, 3, 4$ current at order $\mathcal{O}(x^{4s+2})$.

5.1 Preliminary remarks

As discussed in [16], the 2-CDW contributions are obtained by introducing the 2-words Schottky group elements

$$\gamma = \gamma_{[m_1, m_2]; [m_3, m_4]} = M^{m_1} T^{-1} M^{m_1 - m_2} T M^{-m_2 + m_3} T^{-1} M^{m_3 - m_4} T M^{-m_4}, \quad (5.1)$$

and computing the associated eigenvalue $q_\gamma^{-1/2}$ from (3.7). The sum over 2-CDW words is performed by summing over $\{m_i\}$ in the range

$$0 \leq m_i \leq n-1, \quad m_1 \neq m_2, \quad m_3 \neq m_4, \quad m_2 \neq m_3, \quad m_4 \neq m_1, \quad (m_1, m_2) \neq (m_3, m_4), \quad (5.2)$$

with a global over counting factor $1/2$. An explicit calculation gives the expansion

$$\begin{aligned} q_\gamma = & \left(\frac{x}{4}\right)^4 \frac{1}{s_{m_1-m_2}^2 s_{m_2-m_3}^2 s_{m_1-m_4}^2 s_{m_3-m_4}^2} \\ & + \left(\frac{x}{4}\right)^5 \frac{c_{2m_1-2m_2+2m_3-2m_4} + 8s_{m_1-m_2} s_{m_2-m_3} s_{m_1-m_4} s_{m_3-m_4} - 1}{s_{m_1-m_2}^3 s_{m_2-m_3}^3 s_{m_1-m_4}^3 s_{m_3-m_4}^3} \\ & + \left(\frac{x}{4}\right)^6 \frac{1}{8s_{m_1-m_2}^4 s_{m_2-m_3}^4 s_{m_1-m_4}^4 s_{m_3-m_4}^4} \left(3c_{4m_1-4m_2} - 32c_{2m_1-2m_2} + 3c_{4m_1-4m_3} + 3c_{4m_2-4m_3} \right. \\ & + 32c_{2m_1-2m_3} - 6c_{2(m_1+m_2-2m_3)} - 32c_{2m_2-2m_3} + 6c_{2(m_1-2m_2+m_3)} - 6c_{4m_1-2(m_2+m_3)} + 3c_{4m_1-4m_4} \\ & + 3c_{4m_2-4m_4} + 3c_{4m_3-4m_4} - 32c_{2m_1-2m_4} - 6c_{2(m_1+m_2-2m_4)} + 32c_{2m_2-2m_4} + 6c_{2(m_1+m_3-2m_4)} \\ & - 6c_{2(m_2+m_3-2m_4)} - 32c_{2m_3-2m_4} + 12c_{2(m_1+m_2-m_3-m_4)} + 8c_{2(m_1-m_2+m_3-m_4)} - 6c_{2(m_1-2m_2+m_4)} \\ & - 6c_{2(m_1-2m_3+m_4)} + 6c_{2(m_2-2m_3+m_4)} + 12c_{2(m_1-m_2-m_3+m_4)} + 6c_{4m_1-2(m_2+m_4)} - 6c_{4m_1-2(m_3+m_4)} \\ & \left. - 6c_{4m_2-2(m_3+m_4)} + 38 \right) + \mathcal{O}(x^7), \end{aligned} \quad (5.3)$$

where

$$s_m = \sin \frac{\pi m}{n}, \quad c_m = \cos \frac{\pi m}{n}. \quad (5.4)$$

Unfortunately, the structure of the dependence on the four integers m_i is now quite complicated and no simple regularity is observed in the higher terms. Hence, we cannot provide an all-order expression of q_γ neither in the $n \rightarrow 1$ limit.

6 The leading 2-CDW term

The leading contribution to the entanglement entropy is

$$S_{2\text{-CDW}}^{(s) \text{ one-loop}} = - \lim_{n \rightarrow 1} \frac{1}{n-1} \frac{1}{2} \sum_{\mathcal{P}_2} \left(\frac{q_\gamma^s}{1-q_\gamma} + \dots \right), \quad (6.1)$$

where \mathcal{P}_2 denotes the sum over m_i in the above range, and the dots inside the round bracket give a contribution that starts at order x^{8s} . From the expansion of q_γ , we obtain

$$S_{2\text{-CDW}}^{(s)\text{ one-loop}} = -\lim_{n \rightarrow 1} \frac{1}{n-1} \frac{1}{2} \left(\frac{x}{4}\right)^{4s} F_s(n) + \mathcal{O}(x^{4s+1}), \quad (6.2)$$

where

$$F_s(n) = \sum_{\mathcal{P}_2} \frac{1}{\sin^{2s} \left(\frac{\pi(m_1-m_2)}{n} \right) \sin^{2s} \left(\frac{\pi(m_2-m_3)}{n} \right) \sin^{2s} \left(\frac{\pi(m_1-m_4)}{n} \right) \sin^{2s} \left(\frac{\pi(m_3-m_4)}{n} \right)}. \quad (6.3)$$

We want to obtain an explicit formula for this contribution for a generic (integer ≥ 2) value of s . To this aim, we begin by remarking that $F_s(n)$ can be written as

$$F_s(n) = \text{Tr}(H_n(s)^4) - \text{Tr}(H_n(2s)^2), \quad (6.4)$$

where $H_n(s)$ is the $n \times n$ matrix

$$[H_n(s)]_{pq} = \begin{cases} \csc^{2s} \left(\frac{p-q}{n} \right), & p \neq q, \\ 0, & p = q. \end{cases} \quad (6.5)$$

This matrix is circulant and therefore its eigenvalues are known and read

$$\lambda_n^{(p)}(s) = \sum_{k=1}^{n-1} \cos \left(\frac{2\pi p k}{n} \right) \frac{1}{\sin^{2s} \left(\frac{\pi k}{n} \right)}, \quad p = 0, \dots, n-1. \quad (6.6)$$

Thus,

$$F_s(n) = \sum_{p=0}^{n-1} \left[\lambda_n^{(p)}(s) \right]^4 - \sum_{p=0}^{n-1} \left[\lambda_n^{(p)}(2s) \right]^2. \quad (6.7)$$

At fixed integer s , the eigenvalues $\lambda_n^{(p)}(s)$ are polynomials in both p and n . We checked that the $n \rightarrow 1$ limit of $1/(n-1)$ times $F_s(n)$ can be computed by setting $n = 1$ in those polynomials. The cancelling factor $n-1$ comes entirely from the sum over p . Hence, inside $F_s(n)$, we can exploit the analytically continued expression

$$\lambda_{n \rightarrow 1}^{(p)}(s) = -\frac{(-4)^s}{\Gamma(2s+1)} (p-s)_{2s}, \quad (6.8)$$

where $(a)_b$ is the Pochhammer symbol. This is a finite polynomial and the final step requires the following formula

$$\lim_{n \rightarrow 1} \frac{1}{n-1} \sum_{p=0}^{n-1} p^N = -N \zeta(1-N). \quad (6.9)$$

In conclusion,

$$S_{2\text{-CDW}}^{(s)\text{ one-loop}} = \sigma_s x^{4s} + \mathcal{O}(x^{4s+1}), \quad (6.10)$$

where the coefficient σ_s is obtained from

$$\sigma_s = -\frac{1}{2} \left[\frac{1}{\Gamma(2s+1)^4} (p-s)_{2s}^4 - \frac{1}{\Gamma(4s+1)^2} (p-2s)_{4s}^2 \right]_{p^N \rightarrow -N\zeta(1-N)}, \quad (6.11)$$

as mentioned in the Introduction. Just to give a check of this formula, let us consider the three cases $s = 2, 3, 4$. For $s = 2$, we find

$$F_2(n) = \frac{4n(n^2-4)(n^2-1)}{488462349375} (5703n^{12} + 192735n^{10} + 3812146n^8 + 75493430n^6 + 1249638099n^4 + 9895897835n^2 - 162763727948), \quad (6.12)$$

in agreement with [16]. The evaluation of (6.2) agrees with the coefficient (6.11). For $s = 3$ and $s = 4$, we find

$$F_3(n) = \frac{2n(n^2-4)(n^2-1)}{3028793579456347828125} (7121028183n^{20} + 318929181915n^{18} + 7383843717075n^{16} + 120385864634215n^{14} + 1646919142515245n^{12} + 23516641325405865n^{10} + 379716725855421985n^8 + 6695285026854147965n^6 + 98177254369909711080n^4 + 509870981187643230040n^2 - 18438453061190079393568), \quad (6.13)$$

$$F_4(n) = \frac{4n(n^2-4)(n^2-1)}{7217620475953080409439269921875} (890779990098075n^{28} + 51377796296763575n^{26} + 1506538876909209643n^{24} + 30021482057949152715n^{22} + 460566249653162829602n^{20} + 5915955950696734741150n^{18} + 69830671794873569438670n^{16} + 879907717975387768404750n^{14} + 13575412535055352666915815n^{12} + 253558442419836251074578075n^{10} + 4969225821536517978217020295n^8 + 87557592432594348779328719975n^6 + 1059813722330022285316966964892n^4 + 3656969159093966862988863639760n^2 - 232004722704441042012231546476992). \quad (6.14)$$

and, again, (6.2) agrees (6.11). Clearly, the formula sigma is a very convenient way of computing σ_s at any s since its evaluation is almost instantaneous on any algebraic manipulation system, even for large s .

If one is interested in the entanglement entropy of a scalar field at a certain Δ , the calculation is similar and the leading contribution is easily obtained from the formula (6.11). A CFT calculation would require the analysis of operators quartic in the scalar operator $X(z, \bar{z})$.

7 The complete entanglement entropy of a spin $s = 2, 3, 4$ current at $\mathcal{O}(x^{4s+2})$

To summarise, we present the first three 2-CDW contributions to the cases $s = 2, 3, 4$. The leading term is computed using (6.11). The other two are determined by evaluating the sums over \mathcal{P}_2 as specific polynomials in n and taking the $n \rightarrow 1$ limit in the definition of the entanglement entropy. The result is

$$S_{2\text{-CDW}}^{(s=2)\text{one-loop}} = -\frac{29x^8}{510510} - \frac{100x^9}{2909907} - \frac{14x^{10}}{138567} + \mathcal{O}(x^{11}), \quad (7.1)$$

$$S_{2\text{-CDW}}^{(s=3)\text{ one-loop}} = -\frac{313x^{12}}{148728580} - \frac{392x^{13}}{239028075} - \frac{29321x^{14}}{4621209450} + \mathcal{O}(x^{15}), \quad (7.2)$$

$$S_{2\text{-CDW}}^{(s=4)\text{ one-loop}} = -\frac{113393x^{16}}{1289317436550} - \frac{648x^{17}}{8796512725} - \frac{444896x^{18}}{1236789689135} + \mathcal{O}(x^{19}). \quad (7.3)$$

These contributions must be added to the $S_{\text{CDW,(I)}}^{(s)\text{ one-loop}}$ and $S_{\text{CDW,(II)}}^{(s)\text{ one-loop}}$ terms that we computed previously and that read

$$S_{\text{CDW,(I)}}^{(s=2)\text{ one-loop}} = -\frac{x^4}{630} - \frac{2x^5}{693} - \frac{15x^6}{4004} - \frac{x^7}{234} - \frac{7x^8}{1530} - \frac{84x^9}{17765} - \frac{x^{10}}{209} + \mathcal{O}(x^{11}), \quad (7.4)$$

$$S_{\text{CDW,(II)}}^{(s=2)\text{ one-loop}} = -\frac{x^8}{437580} - \frac{2x^9}{230945} - \frac{x^{10}}{51051} + \mathcal{O}(x^{11}), \quad (7.5)$$

$$S_{\text{CDW,(I)}}^{(s=3)\text{ one-loop}} = -\frac{x^6}{12012} - \frac{x^7}{4290} - \frac{7x^8}{16830} - \frac{28x^9}{46189} - \frac{15x^{10}}{19019} - \frac{2x^{11}}{2093} - \frac{33x^{12}}{29900} - \frac{11x^{13}}{8925} - \frac{143x^{14}}{106488} + \mathcal{O}(x^{15}), \quad (7.6)$$

$$S_{\text{CDW,(II)}}^{(s=3)\text{ one-loop}} = -\frac{x^{12}}{135207800} - \frac{x^{13}}{23401350} - \frac{3x^{14}}{21544100} + \mathcal{O}(x^{15}), \quad (7.7)$$

$$S_{\text{CDW,(I)}}^{(s=4)\text{ one-loop}} = -\frac{x^8}{218790} - \frac{4x^9}{230945} - \frac{3x^{10}}{76076} - \frac{5x^{11}}{71162} - \frac{11x^{12}}{101660} - \frac{11x^{13}}{72675} - \frac{1001x^{14}}{5058180} - \frac{143x^{15}}{580754} - \frac{65x^{16}}{220286} - \frac{104x^{17}}{302841} - \frac{221x^{18}}{565915} + \mathcal{O}(x^{19}), \quad (7.8)$$

$$S_{\text{CDW,(II)}}^{(s=4)\text{ one-loop}} = -\frac{x^{16}}{39671305740} - \frac{4x^{17}}{20419054425} - \frac{2x^{18}}{2398428615} + \mathcal{O}(x^{19}). \quad (7.9)$$

In conclusion, the one-loop entanglement entropies for $s = 2, 3, 4$ at order $\mathcal{O}(x^{4s+2})$ are

$$S^{(s=2)\text{ one-loop}} = -\frac{x^4}{630} - \frac{2x^5}{693} - \frac{15x^6}{4004} - \frac{x^7}{234} - \frac{167x^8}{36036} - \frac{69422x^9}{14549535} - \frac{122x^{10}}{24871} + \mathcal{O}(x^{11}), \quad (7.10)$$

$$S^{(s=3)\text{ one-loop}} = -\frac{x^6}{12012} - \frac{x^7}{4290} - \frac{7x^8}{16830} - \frac{28x^9}{46189} - \frac{15x^{10}}{19019} - \frac{2x^{11}}{2093} - \frac{1644627x^{12}}{1487285800} - \frac{458893x^{13}}{371821450} - \frac{224484047x^{14}}{166363540200} + \mathcal{O}(x^{15}), \quad (7.11)$$

$$S^{(s=4)\text{ one-loop}} = -\frac{x^8}{218790} - \frac{4x^9}{230945} - \frac{3x^{10}}{76076} - \frac{5x^{11}}{71162} - \frac{11x^{12}}{101660} - \frac{11x^{13}}{72675} - \frac{1001x^{14}}{5058180} - \frac{143x^{15}}{580754} - \frac{253702367x^{16}}{859544957700} - \frac{1550029508x^{17}}{4512611027925} - \frac{207186247x^{18}}{530052723915} + \mathcal{O}(x^{19}), \quad (7.12)$$

which are the results summarised in the Introduction.

8 Conclusions

In this paper, we have studied the one-loop correction to the holographic entanglement entropy in AdS_3/CFT_2 . We have considered the specific contribution due to a bulk higher spin s current or a scalar field with full scaling dimension Δ . The case of the scalar field

is an important ingredient of Vasiliev higher spin gravity theory holographically dual to large N conformal coset models. In all cases, we focused on the non-trivial two-interval case at small cross ratio x where a perturbative analysis can be performed. Our main concern has been the $n \rightarrow 1$ entanglement limit of the n -th order Rényi entropy. In this limit, various simplifications occur already on the gravity side of AdS/CFT. In particular, the CDW contribution to the entanglement entropy can be obtained in closed form for the higher spin current at least for the contributions of the form $\mathcal{O}(x^{2s+p})$ up to $\mathcal{O}(x^{4s})$, where 2-CDW elements have to be considered. In the scalar field case, we have computed the similar contributions for generic values of Δ . As a check of the calculation, we have compared the terms up to $\mathcal{O}(x^{2\Delta+5})$ with an explicit CFT calculation finding full agreement. This is clearly not surprising, but the calculation is nevertheless interesting. Indeed, in the entanglement limit, there are simplifications on the CFT side too. In particular, one can check that the set of relevant operators is quite restricted. It is not necessary to take into account the quasi-primaries built with the energy-momentum tensor. This is related to the important fact that scalars do not alter the leading semiclassical entropy that depends universally only on the central charge. At the next-to-leading one-loop order, the contributions from the various bulk fields are decoupled this is the reason behind the simplicity of our CFT comparison. We have also analysed the corrections to the entanglement entropy from the 2-CDW contributions. In particular, we have found an explicit formula for the leading $\mathcal{O}(x^{4s})$ correction as a function of s .

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